Overview

We have seen how a simple divide-and-conquer method can be used to draw curves using three control points. It turns out that the same procedure can be used for curves generated by four control points. We just have a few more steps – and the result this time is a cubic curve.

The Subdivision Procedure

This time our curve is defined by four control points $P_0, P_1, P_2$ and $P_3$. Normally, these points can be arbitrarily placed in three-dimensional space, creating three-dimensional curves. But for illustration purposes on these notes, we will limit ourselves again to drawing in two-dimensions. You should see how this one works quickly.

With the control points $P_0, P_1, P_2$ and $P_3$ the curve will pass through the points $P_0$ and $P_3$ and
will be “influenced” by the points $P_1$ and $P_2$. Again, $P_1$ and $P_2$ are generally “off” the curve, and only lie on the curve in very specific cases (i.e., when the curve is a straight line).

Our divide and conquer procedure subdivides the curve into two segments, each of which is again specified by four control points. With this procedure, we can recursively generate many small segments of the curve, which can be eventually approximated by straight lines when the curve is to be drawn. Again, the most complicated mathematics being the calculation of midpoints of the lines connecting control points.

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**The Basic Subdivision Procedure**

The procedure to subdivide the curve into two segments can be described as follows:

- First, let $P^{(1)}_1$ be the midpoint of the segment $P_0P_1$, let $P^{(1)}_2$ be the midpoint of segment $P_1P_2$, and let $P^{(1)}_3$ be the midpoint of segment $P_2P_3$.

\[ \begin{align*}
P^{(1)}_1 & \quad \text{midpoint of } P_0P_1, \\
P^{(1)}_2 & \quad \text{midpoint of } P_1P_2, \\
P^{(1)}_3 & \quad \text{midpoint of } P_2P_3.
\end{align*} \]

- Now, let $P^{(2)}_2$ be the midpoint of the segment $P^{(1)}_1P^{(1)}_2$, and let $P^{(2)}_3$ be the midpoint of segment $P^{(2)}_1P^{(2)}_2$.

\[ \begin{align*}
P^{(2)}_2 & \quad \text{midpoint of } P^{(1)}_1P^{(1)}_2, \\
P^{(2)}_3 & \quad \text{midpoint of } P^{(2)}_1P^{(2)}_2.
\end{align*} \]
• and finally let $P_3^{(2)}$ be the midpoint of the segment $P_2^{(2)}P_3^{(2)}$.

We define $P_3^{(3)}$ to be a point on the curve.

Same procedure! Right? Same notation, but one level deeper. Here’s what we have done...

• We have generated another point on the curve – $P_3^{(3)}$. We already had two points on the curve, $P_0$ and $P_3$, and now we have a third.
We have effectively subdivided the curve into two segments. The first segment has control points \( \{ P_0, P_1^{(1)}, P_2^{(2)}, P_3^{(3)} \} \) (note that the beginning and end points are both on the curve, and the middle two points are “influencing” the curve.). The second segment has control points \( \{ P_3^{(3)}, P_3^{(2)}, P_3^{(1)}, P_3 \} \).

So, now just continue the divide-and-conquer process as we did with the quadratic curve.

Summary
Again, this is a geometric method, using only the midpoint formula as its fundamental tool. It uses the basic computer science paradigm of (sub)divide and conquer to calculate points on the curve. The curve can be “drawn” using computer graphics by calculating a somewhat-dense set of points, and connecting them with straight lines.