Overview

A general method can be specified to subdivide a Bézier patch. This method is specified unlike the matrix methods, as it is based upon the definition of the patch as a set of curves.

The Method for Subdivision

We recall that, if we take the analytic equation of a Bézier patch, fix $u$ and group factors appropriately, we obtain

$$P(u, v) = \sum_{j=0}^{m} \left[ \sum_{i=0}^{n} P_{i,j} B_{i,n}(u) \right] B_{j,m}(v)$$

We notice that portion of the equation inside the brackets is the representation of a Bézier curve. If we let $Q_j(u)$ be the value inside the brackets, i.e.

$$Q_j(u) = \sum_{i=0}^{n} P_{i,j} B_{i,n}(u)$$

Then

$$P(u, v) = \sum_{j=0}^{m} Q_j(u) B_{j,m}(v)$$

That is, the quantities $Q_j(u)$ form the control points of another Bézier curve, and together for all $u$ and $v$, they form the surface.

If, then, we subdivide each of the $m$ rows of the $P_{i,j}$ matrix, it implies that the $Q_j$'s in the above equation represent only points from the first half of the patch (with respect to $u$). The following illustration shows the result of subdividing the rows in the $4 \times 4$ case.
The second half of the patch can be obtained in a similar fashion. The first and second half of
the patch, with respect to $v$, can be obtained by subdividing the columns.

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**Patch Subdivision Using the Matrix Form**

Suppose we wish to subdivide the patch at the point $u = \frac{1}{2}$. We reparameterize the matrix
equation above (by substituting $\frac{u}{2}$ for $u$) to cover only the first half of the patch, and simplify to
obtain.
\( P\left( \frac{u}{2}, v \right) = \begin{bmatrix} 1 & \left( \frac{u}{2} \right)^2 & \left( \frac{u}{2} \right)^3 \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \)

\[
= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\]

where the matrix \( S_L \) is defined as
and is identical to the left subdivision matrix for the curve case. So in particular, the subpatch \( P\left(\frac{4}{2}\right) \) is again a Bézier patch and the quantity

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}
\begin{bmatrix} P_{0,0} \\ P_{1,0} \\ P_{2,0} \\ P_{3,0} \end{bmatrix}
\]

forms the control points of this patch.

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**Calculation of the Second Half of the Patch**

In the same way, we can obtain the subdivision matrix for the second half of the patch. First
we reparameterize the original curve, and then simplify to obtain

\[
P\left(\frac{1}{2} + \frac{u}{2}, v\right) = \left[ 1 \ (\frac{1}{2} + \frac{u}{2}) \ (\frac{1}{2} + \frac{u}{2})^2 \ (\frac{1}{2} + \frac{u}{2})^3 \right] M \begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix} M^T \begin{bmatrix}
1 \\
v \\
v^2 \\
v^3
\end{bmatrix}
\]

\[
= \left[ 1 \ u \ u^2 \ u^3 \right] M S_R \begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix} M^T \begin{bmatrix}
1 \\
v \\
v^2 \\
v^3
\end{bmatrix}
\]

where

\[
S_R = \begin{bmatrix}
\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

which is identical to the right subdivision matrix in the curve case and we obtain a matrix that can be applied to a set of control points to produce the control points for the second half of the patch.

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**General Subdivision with either Parameter**

We can develop a procedure to generate the control points for the first and second portions of the patch when subdivision is done with respect to \( v \). These are

\[ PS_L \text{ and } PS_R \]
where

\[
P = \begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix}
\]

The development is exactly parallel to that with respect to \( u \).

Combining these two methods, we can see that the arrays

\[
S_L P S_L \\
S_R P S_L \\
S_L P S_R \\
S_R P S_R
\]

segment the patch into quarters, the first array being the quarter that corresponds to \( 0 \leq u \leq \frac{1}{2}, 0 \leq v \leq \frac{1}{2} \), the second to the one that corresponds to \( 0 \leq u \leq \frac{1}{2}, \frac{1}{2} \leq v \leq 1 \), etc.

**Summary**

So, using only curve methods, and by subdividing the rows or columns of the control point array, we can effectively subdivide a Bézier patch. This is the most frequently used algorithm in software implementations of subdivision and can be utilized for Bézier patches of arbitrary degree.