Overview

What is the Bézier curve was not parameterized between 0 and 1? How would it be written? We answer these questions here, and it is easy.

Modifying the Definition of a Bezier Curve

If we look at the geometric definition of a Bézier curve

\[ P(t) = P_{n-1}(t) \]

where

\[ P_i^{(j)}(t) = \begin{cases} (1-t)P_{i-1}^{(j-1)}(t) + tP_i^{(j-1)}(t) & \text{if } j > 0, \\ P_i & \text{otherwise} \end{cases} \]

where \( t \) ranges between zero and one \( 0 \leq t \leq 1 \). It is easy to see how to generalize the curve. In particular, the each of the quantities

\[ (1-t)P_{i-1}^{(j-1)}(t) + tP_i^{(j-1)}(t) \]

(1)
gives a point on the line \( P_{i-1}^{(j-1)}(t)P_i^{(j-1)}(t) \), and this point is given by the parameter \( t \). (if \( t = \frac{1}{2} \), it is the midpoint of the line.). We can generalize this by noticing that Equation (1) is just an affine combination, and we can get other points on a line by replacing the \( t \) and \( (1-t) \) terms of Equation (1) by any two functions that always sum to one. That is, we could replace it by

\[ \alpha_1(t)P_{i-1}^{(j-1)}(t) + \alpha_2(t)P_i^{(j-1)}(t) \]

where \( \alpha_1(t) + \alpha_2(t) = 1 \). Now, in general, we won’t get a Bézier curve, but in many cases, we can
substitute meaningful functions that do define interesting curves. Since we are always selecting a point on the line connecting two points, and if we can guarantee that $0 \leq \alpha_1, \alpha_2 \leq 1$, our curve will always lie within the convex hull of the control points.

Modifying the Parameterization of the Bézier Curve

If we look at the geometric definition of a Bézier curve

$$\mathbf{P}(t) = \mathbf{P}_{n-1}^{(n-1)}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-t)\mathbf{P}_{i-1}^{(j-1)}(t) + t\mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0, \\ \mathbf{P}_i & \text{otherwise} \end{cases}$$

where $t$ ranges between zero and one $0 \leq t \leq 1$, this definition is restricted for $t$. What is we wanted $t$ to lie in a different interval?

The solution is straightforward. If we want $t \in [a, b]$, then we write

$$\mathbf{P}(t) = \mathbf{P}_{n-1}^{(n-1)}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} \frac{b-t}{b-a} \mathbf{P}_{i-1}^{(j-1)}(t) + \frac{t-a}{b-a} \mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0, \\ \mathbf{P}_i & \text{otherwise} \end{cases}$$

where $t \in [a, b]$.

We note that

$$\frac{b-t}{b-a} + \frac{t-a}{b-a} = \frac{(b-t) + (t-a)}{b-a} = \frac{b-a}{b-a} = 1$$

so we still have an affine combination. Note that if $a = 0$ and $b = 1$, we just have the original equation.

It is useful to introduce a function $\tau(t)$ into the definition. This simplifies things somewhat, and
will help us better analyze this curve. So, we rewrite the definition as

$$P(t) = P_{n-1}^{(n-1)}(t)$$

where

$$P^{(j)}_i(t) = \begin{cases} (1 - \tau(t))P^{(j-1)}_{i-1}(t) + \tau(t)P^{(j-1)}_i(t) & \text{if } j > 0, \\ P_i & \text{otherwise} \end{cases}$$

where $\tau(t) = \frac{t-a}{b-a}$ and $t \in [a, b]$.

Thus, we can parameterize our curve so that $t$ lies in an arbitrary interval $[a, b]$.

Summary

Reparameterization is straightforward for the Bézier curve. Now we can start piecing these curves together.