Piecing Two Bézier Curves Together

Given two Bézier curves \( P(t) \) and \( Q(t) \) of the same degree \( n \), defined by two sets of control points \( \{P_0, P_1, ..., P_n\} \) and \( \{Q_0, Q_1, ..., Q_n\} \), there are some immediate observations one can make when piecing these curves together. First, we must choose a parameterization so that the resulting curve, say \( R(t) \), has a smooth variation of parameters. For example, let’s assume that \( P(t) \) is parameterized in the interval \([a, b]\), and \( Q(t) \) is parameterized in the interval \([b, c]\). Then \( R(t) \) will be parameterized in the interval \([a, c]\).

The figure below shows two Bézier curves pieced together. Here we have used \( a = 0 \), \( b = 1 \), and \( c = 2 \).

In order to insure that the curve \( R(t) \) is continuous, \( i.e., C^0 \), we must insure that \( P(b) = Q(b) \). That is, \( P_n = Q_0 \).
In order to insure that the curve $R$ is $C^1$, we must have that the derivative is continuous. Since the individual Bézier curve components are automatically $C^1$ (They are polynomials, which are $C^\infty$, all derivatives are continuous), we just need to insure that $R'(t)$ is continuous at $b$. This implies that $P'(b) = Q'(b)$, and calculating derivatives, we have

\[
P'(b) = \frac{n}{b-a} (P_n - P_{n-1})
\]
\[
Q'(b) = \frac{n}{c-b} (Q_1 - Q_0)
\]

Substituting for $P_n$ and $Q_0$, which are the same number, we have

\[
\frac{n}{c-b} (Q_1 - R(b)) = \frac{n}{b-a} (R(b) - P_{n-1})
\]

and simplifying, we get

\[
Q_1 = \frac{c-b}{b-a} (R(b) - P_{n-1}) + R(b)
\]

which says that $Q_1$ must lie on the line $P_{n-1}P_n$, and be a fixed distance from $R(b) = P_n = Q_0$. So in order to get $C^1$, $Q_1$ is determined by the parameterization $a$, $b$, and $c$, and the two points $P_{n-1}$ and $P_n$. Interesting!!

In the illustration above, where $a = 0$, $b = 1$, and $c = 2$, the constant $\frac{c-b}{b-a}$ is just 1, and we can see that $Q_1$ not only lies on the line $P_2P_3$, but the vectors $P_3 - P_2$ and $Q_1 - Q_0$ are the same.

What if we want the curve to be $C^2$?

Then the second derivatives at $b$ must match, i.e., $R''(b) = P''(b) = Q''(b)$. Differentiating the analytic definition of the Bézier curve gives

\[
P'(t) = n \sum_{i=0}^{n} P_i [B_{i-1,n-1}(t) - B_{i,n-1}(t)]
\]
\[
= n \sum_{i=0}^{n-1} [P_{i+1} - P_i] B_{i,n-1}(t)
\]
and differentiating again, we obtain

\[ P''(t) = n(n - 1) \sum_{i=0}^{n-2} \left[(P_{i+2} - P_{i+1}) - (P_{i+1} - P_i)\right] B_{i,n-2}(t) \]

\[ = n(n - 1) \sum_{i=0}^{n-2} [P_{i+2} - 2P_{i+1} + P_i] B_{i,n-2}(t) \]

and if we add in the parameterization, using the chain rule, we obtain

\[ P'(\frac{t-a}{b-a}) = n\frac{1}{b-a} \sum_{i=0}^{n-2} [P_{i+1} - P_i] B_{i,n-1}(\frac{t-a}{b-a}), \text{ and} \]

\[ P''(\frac{t-a}{b-a}) = n(n - 1) \frac{1}{(b-a)^2} \sum_{i=0}^{n-2} [P_{i+2} - 2P_{i+1} + P_i] B_{i,n-2}(\frac{t-a}{b-a}) \]

Similarly,

\[ Q'(\frac{t-b}{c-b}) = n\frac{1}{b-a} \sum_{i=0}^{n-2} [Q_{i+1} - Q_i] B_{i,n-1}(\frac{t-b}{c-b}), \text{ and} \]

\[ Q''(\frac{t-b}{c-b}) = n(n - 1) \frac{1}{(b-a)^2} \sum_{i=0}^{n-2} [Q_{i+2} - 2Q_{i+1} + Q_i] B_{i,n-2}(\frac{t-b}{c-b}) \]

Thus if \( P''(b) = Q''(b) \), we must have

\[ \frac{1}{(b-a)^2} [P_n - 2P_{n-1} + P_{n-2}] = \frac{1}{(c-b)^2} [Q_0 - 2Q_1 + Q_2] \]

So, since \( P_n = Q_0 \), and \( Q_1 \) is determined by Equation (1), we can solve directly for \( Q_2 \).

Thus, if we desire \( C^2 \) for the curve \( R \), then all of \( Q_0 \), \( Q_1 \), and \( Q_2 \) are determined by the positions of \( P_{n-2}, P_{n-1}, \) and \( P_n \), and the parameterization we choose for the curve. So only \( Q_3, Q_4, ..., Q_n \) are free to choose.

Thus, in the illustration below where we are trying to piece together two cubic Bézier curves \( P(t) \) and \( Q(t) \) with parameterizations \([0, 1]\) and \([1, 2]\), respectively, we are really only able to choose \( Q_3 \) freely if we want the curve to be \( C^2 \).
If we desire $C^3$ continuity on the illustrated curve case above, then all control points $Q_0$, $Q_1$, $Q_2$, and $Q_3$ will be determined by the control points $P_0$, $P_1$, $P_2$, and $P_3$, and the parameterization we choose. **In this case the two control point sets represent the same Bézier curve.**

**Summary**

Piecing together two Bézier curves with $C^k$ continuity determines many of the control points *a priori.*