Affine Combinations of Points

Points are utilized to position ourselves or objects within our three-dimensional space. The operations on the vectors in our space are numerous – addition, scalar multiplication, dot products, cross products – but the operations on the points are limited. In this section we discuss the basic operation on points – affine combinations.

Affine Combinations of Two Points

Let \( P_1 \) and \( P_2 \) be points, and consider the expression

\[
P = P_1 + t(P_2 - P_1)
\]

This equation is meaningful, as \( P_2 - P_1 \) is a vector, and thus so is \( t(P_2 - P_1) \). Therefore \( P \) is the sum of a point and a vector which is again a point. For each \( t \), the point \( P \) represents a point on the line that passes through \( P_1 \) and \( P_2 \).

We note that if \( 0 \leq t \leq 1 \) then \( P \) is somewhere on the line segment joining \( P_1 \) and \( P_2 \). If \( t > 1 \) then the point \( P \) is still on the line, but to the right of \( P_2 \) in our illustration. Similarly if \( t < 0 \), then \( P \) is to the left of \( P_1 \).

We can now define an affine combination of two points \( P_1 \) and \( P_2 \) to be

\[
P = \alpha_1 P_1 + \alpha_2 P_2
\]
where \( \alpha_1 + \alpha_2 = 1 \). \[ P = (1-t)P_1 + tP_2 \] is shown to be an affine transformation by setting \( \alpha_2 = t \).

Affine Combinations of Three or More Points

We can generalize this to define an affine combination of an arbitrary number of points. If \( P_1, P_2, \ldots, P_n \) are points and \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are scalars such that \( \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1 \), then we can define a new point \( P \) by

\[
P = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_n P_n
\]

which is defined to be

\[
P_1 + \alpha_2(P_2 - P_1) + \cdots + \alpha_n(P_n - P_1)
\]

An Example with Triangles

To construct an excellent example of an affine combination consider three points \( P_1, P_2 \) and \( P_3 \). A new point \( P \) defined by

\[
P = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3
\]

where \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \), gives a point in the triangle \( \triangle P_1 P_2 P_3 \). We note that the definition of affine combination defines this point to be

\[
P = P_1 + \alpha_2(P_2 - P_1) + \alpha_3(P_3 - P_1)
\]

The following illustration shows the point \( P \) generated when \( \alpha_1 = \alpha_2 = \frac{1}{4} \) and \( \alpha_3 = \frac{1}{2} \).
In fact, it can be easily shown that if $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$ then the point $P$ will be within (or on the boundary) of the triangle. If any $\alpha_i$ is less than zero or greater than one, then the point will lie outside the triangle. If any $\alpha_i$ is zero, then the point will lie on the boundary of the triangle.

Affine combinations of points are very important in geometric modeling. It is the only way we can combine points together to obtain another point. They key feature is that the coefficients must sum to one.