Geometric Modeling Notes

PARAMETRIC CURVES

Kenneth I. Joy
Institute for Data Analysis and Visualization
Department of Computer Science
University of California, Davis

Parametric Curves

For most of our academic careers we have dealt with functions and curves that have the form
\[ y = f(x) \]. This form is nice in which to illustrate mathematical concepts, but it is too restrictive
when trying to represent curves and surfaces – for each \( x \), we are given only one \( y \).

In these notes, we present the idea of parametric curves and surfaces, which are used ex-
tensively in computer graphics and geometric modeling. These allow us more flexibility in the
definition of curves and surfaces.

A parametric curve in two dimensions is defined through a parameter \( t \), and has defining
functions for both the \( x \) and \( y \) coordinates.

\[
x = f(t) \\
y = g(t)
\]

where \( t \) is usually restricted to an interval \( t \in [a, b] \).

Here we have one \((x, y)\) pair for each \( t \), with \( x \) and \( y \) varying according to different functions,
and this gives us the flexibility to generate a number of curves.

Frequently a parametric curve is written as \((x(t), y(t))\), emphasizing the two functions that
define the curve, or is written \( \mathbf{P}(t) \), where \( \mathbf{P} \) is a two-dimensional point [You will see this one over
and over, as it simplifies the mathematics]. We just have to remember that the functions exist for
each parameter.

A Straight Line
Given two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, we can form the straight line between them by

$$(x, y) = (1 - t)(x_0, y_0) + t(x_1, y_1)$$

$$= ((1 - t)x_0 + tx_1, (1 - t)y_0 + ty_1)$$

where $t \in [0, 1]$. Here, $f(t) = (1 - t)x_0 + tx_1$, and $g(t) = (1 - t)y_0 + ty_1$ in the definition above.

We will also write this straight line in the form

$$P(t) = (1 - t)P_0 + tP_1$$

using $P = (x, y)$, and recognizing that the math is done to each component of $P$ independently.

---

**A Circle**

We can represent a circle parametrically by

$$x = \cos(t)$$

$$y = \sin(t)$$

where $t \in [0, 2\pi]$.

---

**My Favorite Parametric Curve**

Here is my favorite parametric curve:

$$x = \frac{1 - t^2}{1 + t^2}$$

$$y = \frac{2t}{1 + t^2}$$

where $t \in [0, 1]$.

What curve is this? Well, we can see by substituting $t = 0$, we get the point $(1, 0)$, and by substituting $t = 1$, we get the point $(0, 1)$. So the curve passes through these points. What about
the rest of the curve? Well, it is easiest to see this curve if we calculate $x^2 + y^2$.

$$x^2 + y^2 = \left(\frac{1 - t^2}{1 + t^2}\right)^2 + \left(\frac{2t}{1 + t^2}\right)^2$$

$$= \left(1 - t^2\right)^2 + \left(2t\right)^2$$

$$= \left(1 + t^2\right)^2$$

Thus, we have that $x^2 + y^2 = 1$, and the curve is a **quarter circle**.

This is an unusual representation of a circle – called a *rational* polynomial representation. The functions are just ratios of two quadratic polynomials (parabolas), instead of the *sin* or *cos* representations that we are normally taught. It shows us that parameter curves take many forms.

For parametric representations of surfaces, we use two parameters, say $u$ and $v$, and represent the surface as

$$x = f(u, v)$$

$$y = g(u, v)$$

$$z = h(u, v)$$

for some functions $f$, $g$, and $h$, and where $u$ and $v$ are contained in some intervals, $u \in [a, b]$, $v \in [c, d]$.

We will also use these extensively in our work.

**Summary**

Get used to parametric representations for curves and surfaces. They are used heavily in computer graphics and geometric modeling.