Overview

Bézier curves are approximating curves, in that they do not pass through the control points. However, it is easy to determine a Bézier curve that interpolates a set of points using the matrix definition of a Bézier curve. We will concentrate on cubic Bézier curves, but the techniques scale up to all degrees easily.

Interpolating Bezier Curves

Given a set of control points \( \{P_0, P_1, P_2, P_3\} \), the cubic Bézier curve \( P(t) \) defined by these control points is

\[
P(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}
\]

where

\[
M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}
\]

We know that \( P(0) = P_0 \), and \( P(1) = P_3 \), and if we have two additional points that we want to
curve to interpolate, say \( P(a) = Q_1 \) and \( P(b) = Q_2 \), then we can write

\[
P(a) = \begin{bmatrix} 1 & a & a^2 & a^3 \end{bmatrix} M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}
\]

and

\[
P(b) = \begin{bmatrix} 1 & b & b^2 & b^3 \end{bmatrix} M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}
\]

and putting all these together, we have

\[
\begin{bmatrix} P(0) \\ P(a) \\ P(b) \\ P(1) \end{bmatrix} = \begin{bmatrix} P_0 \\ Q_1 \\ Q_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} M \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}
\]

Therefore, we can define

\[
\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = M^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} P_0 \\ Q_1 \\ Q_2 \\ P_3 \end{bmatrix}
\]

to determine the control points \( P_1 \) and \( P_2 \) that cause the curve to pass through \( Q_1 \) and \( Q_2 \) respectively. The matrix with \( a \)'s and \( b \)'s is always invertable if \( 0 < a < b < 1 \).