Overview

We are working with the quadratic Bézier curve, and have discovered that our initial divide-and-conquer approach can be simplified greatly by introducing a parameter $t$, $0 \leq t \leq 1$, and using this value to help choose points on the curve.

So far, we have used geometric arguments to make our calculations, but now we can develop an analytic formula for our curve, and discover why the curve is quadratic, and that it is really a parabola.

Developing the Equation of the Curve

There is a different way of looking at this procedure – because there is a parameter involved. Each one of the points $P_1^{(1)}$, $P_2^{(1)}$, and $P_2^{(2)}$ is really a function of the parameter $t$ – and $P_2^{(2)}$ can be equated with $P(t)$ since it is a point on the curve that corresponds to the parameter value $t$. In this way, $P(t)$ can be seen as a functional representation of the Bézier curve.

Writing down the algebra, we see that

$$P(t) = P_2^{(2)}(t)$$
$$= (1 - t)P_1^{(1)}(t) + tP_2^{(1)}(t)$$
where

\[ P_1^{(1)}(t) = (1 - t)P_0 + tP_1, \text{ and} \]

\[ P_2^{(1)}(t) = (1 - t)P_1 + tP_2 \]

(Note that we have now denoted \( P_1^{(1)} \) and \( P_2^{(1)} \) as functions of \( t \).) Substituting these two equations back into the original, we have

\[
P(t) = (1 - t)P_1^{(1)}(t) + tP_2^{(1)}(t)
\]

\[
= (1 - t)[(1 - t)P_0 + tP_1] + t[(1 - t)P_1 + tP_2]
\]

\[
= (1 - t)^2P_0 + (1 - t)tP_1 + t(1 - t)P_1 + t^2P_2
\]

\[
= (1 - t)^2P_0 + 2t(1 - t)P_1 + t^2P_2
\]

This is quadratic polynomial (as it is a linear combination of quadratic polynomials), and therefore it is a parabolic segment. Thus, the quadratic Bézier curve is simply a parabolic curve.

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Summary

The quadratic Bézier curve serves as a good example for discussing the development of the Bézier curve, but really only generates parabolas. This eliminates the curve for many applications where smooth curves with inflection points are necessary. However, the neat thing about the Bézier curve is that the curves generated with four or more control points, can be generated in almost the same way as the quadratic curve.