On-Line Visualization Notes

Gradients and Normals

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Overview

If we consider a scalar field in space given by the function \( f(x, y, z) \), we know that the first partial derivatives of \( f \) are the rates of change of \( f \) in the directions of the coordinate axes. The gradient of \( f \), \( \nabla f \) is the direction of maximum rate of change of the function at any point. When generating isosurfaces, this gradient forms the normal vector to the isosurface at any point. This document develops the gradient from directional derivatives, and shows that the gradient to a point can represent the normal to the isosurface through the point.

Directional Derivatives

Given a point \( p \) in space, and a unit direction vector \( \vec{v} \) at \( p \). The directional derivative of \( f \) at \( p \) in the direction \( \vec{v} \), denoted \( D_{\vec{v}} f(p) \) is given by

\[
D_{\vec{v}} f(p) = \lim_{t \to 0} \frac{f(p + t\vec{v}) - f(p)}{t}
\]

This is obviously the rate of change of \( f \) at \( p \) in the direction \( \vec{v} \).

To put this directional derivative in terms of the partials in the directions of the three coordinate axes, we note that the ray \( p + t\vec{v} \) can be written in terms of its three components as

\[
p + t\vec{v} = \begin{pmatrix} x_p + tx_v \\ y_p + ty_v \\ z_p + tz_v \end{pmatrix}
\]
where \( \mathbf{p} = (x_p, y_p, z_p) \) and \( \mathbf{v} = <x_v, y_v, z_v> \). Then
\[
D_{\mathbf{v}}f(\mathbf{p}) = \frac{\partial f}{\partial t}
= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz
= \frac{\partial f}{\partial x} x_v + \frac{\partial f}{\partial y} y_v + \frac{\partial f}{\partial z} z_v
\]
using the chain rule.

**Gradients**

We call the vector
\[
\left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right>
\]
the *gradient* of \( f \), and denote this as \( \nabla f \). It should be clear from the above that the directional derivative of \( f \) at \( \mathbf{p} \) in the direction \( \mathbf{v} \) is given by
\[
D_{\mathbf{v}}f(\mathbf{p}) = \nabla f(\mathbf{p}) \cdot \mathbf{v}
\tag{1}
\]
We note that if the vector \( \mathbf{v} \) is in the direction of the \( x \) axis, then \( D_{\mathbf{v}}f(\mathbf{p}) = \frac{\partial f}{\partial x} \). Similar results hold for the \( y \) and \( z \) axes respectively.

**The Gradient is in the Direction of Maximum Increase**

From Equation (1) we have that
\[
D_{\mathbf{v}}f(\mathbf{p}) = \nabla f(\mathbf{p}) \cdot \mathbf{v}
= |\nabla f(\mathbf{p})||\mathbf{v}| \cos \theta
\]
where \( \theta \) is the angle between the vector \( \mathbf{v} \) and \( \nabla f(\mathbf{p}) \). Since \( |\nabla f(\mathbf{p})| \) is constant, and \( \mathbf{v} \) is a unit vector, we see that this dot product is maximized when \( \cos \theta = 1 \), or when \( \theta = 0 \). This says that the direction of maximal increase of the function – *i.e.*, where \( D_{\mathbf{v}}f(\mathbf{p}) \) is maximized – is when \( \mathbf{v} \) is in the direction of the gradient of \( f \) at \( \mathbf{p} \).

**The Gradient at a Normal to Surfaces**

Consider an *isosurface* of \( f \), *i.e.*, the surface where \( f(\mathbf{p}) = c \) for some constant \( c \). By letting \( c \) assume all values, we obtain a family of surfaces, the *level surfaces* of \( f \). For each point in space, exactly one level surface of \( f \) will pass through the point.
Any curve in space can be written as \( p(t) \) for a parameter \( t \), and if we require this curve to be on the isosurface \( f(p) = c \), we have that

\[
f(p(t)) = c
\]

Differentiating this with respect to \( t \) and using the chain rule, we have that

\[
0 = \frac{\partial}{\partial t} f(p(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}
\]

\[
= \nabla f \cdot p'(t)
\]

Here we have separated \( p(t) \) into its component form, \( p(t) = (x(t), y(t), z(t)) \) and

\[
p'(t) = \left< \frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t) \right>
\]

Therefore, the gradient is perpendicular to the tangent vector of the surface at \( p \), for any curve \( p(t) \) on the surface that passes through \( p \). So the gradient must be in the direction of the normal to the surface.

**A Two-Dimensional Example**

Consider the scalar field given by the function

\[
f(x, y) = x^2 + y^2
\]

which has a set of circles as the level curves of the field. The gradient of \( f \) is equal to

\[
\nabla f(x, y) = < 2x, 2y >
\]

Given a point \((x, y)\) on the circle \( f(x, y) = r \), we know by polar coordinates that \((x, y) = (r \cos(t), r \sin(t))\). The tangent vector \( \vec{t} \) to this curve at \((x, y)\) is \(< -r \sin(t), r \cos(t) >\) (just differentiate with respect to \( t \)), which is equal to \(< -y, x >\).

We can see then that the gradient is perpendicular to the tangent vector since

\[
\nabla f(x, y) \cdot \vec{t} = 2x \cdot (-y) + 2y \cdot x = 0
\]

Thus, the gradient is normal to the curve.
Summary

The gradient of a scalar field $f$ is an important concept in visualization. The gradient of $f$ represents the direction of maximum change in the function $f$, and can be used as the normal to level surfaces of the scalar field.